Massive relic gravitational waves from f(R) theories of gravity: production and potential detection

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Abstract

The production of a stochastic background of relic gravitational waves is well known in various works in the literature, where, by using the so called adiabatically-amplified zero-point fluctuations process, it has been shown how the standard inflationary scenario for the early universe can in principle provide a distinctive spectrum of relic gravitational waves. In this paper, it is shown that, in general, f(R) theories of gravity produce a third massive polarization of gravitational waves and the primordial production of this polarization is analysed adapting the adiabatically-amplified zero-point fluctuations process at this case. In this way, previous results, where only particular cases of f(R) theories have been analysed, will be generalized.

The presence of the mass could also have important applications in cosmology, because the fact that gravitational waves can have mass could give a contribution to the dark matter of the Universe.

An upper bound for these relic gravitational waves, which arises from the WMAP constrains, is also released.

At the end of the paper, the potential detection of such massive gravitational waves using interferometers like Virgo and LIGO is discussed.

1 Introduction

Recently, the data analysis of interferometric gravitational waves (GWs) detectors has been started (for the current status of GWs interferometers see [1, 2, 3, 4, 5, 6, 7, 8]) and the scientific community aims in a first direct detection of GWs in next years.

Detectors for GWs will be important for a better knowledge of the Universe and also to confirm or to rule out the physical consistency of General Relativity or of any other theory of gravitation [9, 10, 11, 12, 13, 14]. In fact, in the context of Extended Theories of Gravity, some differences between General Relativity and the others theories can be pointed out starting by the linearized theory of gravity [9, 10, 12, 14].

In this picture, detectors for GWs are in principle sensitive also to a hypothetical *scalar* component of gravitational radiation, that appears in extended theories of gravity like scalar-tensor gravity and high order theories [12, 15, 16, 17, 18, 19, 20, 21, 22], Brans-Dicke theory [23] and string theory [24].

A possible target of these experiments is the so called stochastic background of gravitational waves [25, 26, 27, 28, 29, 30].

The production of the primordial part of this stochastic background (relic GWs) is well known in the literature starting by the works of [25, 26] and [27, 28], that, using the so called adiabatically-amplified zero-point fluctuations process, have shown in two different ways how the standard inflationary scenario for the early universe can in principle provide a distinctive spectrum of relic gravitational waves. In [29, 30] the primordial production has been analysed for the scalar component admitted from scalar-tensor gravity.

In this paper, it is shown that, in general, f(R) theories of gravity produce a third massive polarization of gravitational waves and the primordial production of this polarization is analysed adapting the adiabatically-amplified zero-point fluctuations process at this case. In a recent paper [21], such a process has been applied to the same class of theories, i.e. the $f(R)$ ones, which will be discussed in the present work. But, in [21] a different point of view has been considered. In that case, by using a conform analysis, the authors discussed such a process in respect to the two standard polarizations which arises from standard General Relativity. In the present paper the analysis is focused to the third massive polarization. In this way, we generalize previous results where only particular cases of f(R) theories have been analysed (see [40] for example).

Regarding f(R) theories, even if such theories have to be dismissed either because they contradict the current cosmology and solar system tests or generate "ghosts" in what would be a quantum version of the finite range gravity[41], a number of authors manage to avoid both of such problems [41].

Concerning the presence of the graviton mass, it was considered by many authors since Fierz and Pauli (1939) [42, 43]. Important contributions in this sense are the ones in [44, 45, 46]. In these papers, it is emphasized that the presence of the mass modifies General Relativity on various scales, which are defined by the Compton wavelength of gravitons. This fact leads to dispersion of waves and Yukawa-like potentials in the linearized theory. On the other hand, it also leads to other modifications in General Relativity solutions in high-order (i.e. non-linear) regimes. The fact that gravitational waves can have mass could also have important applications in cosmology because such masses could give a contribution to the dark matter of the Universe.

An upper bound for these relic gravitational waves, which arises from the

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WMAP constrains, is also released. About this point, a different interesting treatment of the additional polarizations and impact on Cosmic Microwave Background has been recently discussed in the good paper [47].

At the end of the paper the potential detection of such massive gravitational waves using interferometers like Virgo and LIGO is discussed.

2 f(R) theories of gravity

Let us consider the action

$$S = \int d^4x \sqrt{-g} f(R) + \mathcal{L}_m, \tag{1}$$

where R is the Ricci curvature scalar.

Equation (1) represents the action of the so called $f(R)$ theories of gravity [9, 10, 11, 19, 21, 29, 33, 34, 35] in respect to the well known canonical one of General Relativity (the Einstein - Hilbert action [31, 32]) which is

$$S = \int d^4x \sqrt{-g} R + \mathcal{L}_m. \tag{2}$$

The action (1) has been analysed in [33, 34, 35] in cosmological contexts.

As we will interact with gravitational waves, i.e. the linearized theory in vacuum, $\mathcal{L}_m = 0$ will be put and the pure curvature action

$$S = \int d^4x \sqrt{-g} f(R) \tag{3}$$

will be considered.

3 Field equations and linearized theory

Following [32] (note that in this paper we work with $8\pi G = 1$, $c = 1$ and $\hbar = 1$), the variational principle

$$\delta \int d^4x \sqrt{-g} f(R) = 0 \tag{4}$$

in a local Lorentz frame can be used, obtaining:

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - f'(R)_{;\mu;\nu} + g_{\mu\nu}\Box f'(R) = 0 \tag{5}$$

which are the modified Einstein field equations. $f'(R)$ is the derivative of f in respect to the Ricci scalar. Writing down, explicitly, the Einstein tensor, eqs. (5) become

$$G_{\mu\nu} = \frac{1}{f'(R)}\{\frac{1}{2}g_{\mu\nu}[f(R) - f'(R)R] + f'(R)_{;\mu;\nu} - g_{\mu\nu}\Box f'(R)\}. \tag{6}$$

The trace of the field equations (6) gives

$$3\Box f'(R) + Rf'(R) - 2f(R) = 0, \tag{7}$$

and, with the identifications [40]

$$\Phi \to f'(R) \qquad \text{and} \qquad \frac{dV}{d\Phi} \to \frac{2f(R) - Rf'(R)}{3} \tag{8}$$

a Klein - Gordon equation for the effective Φ scalar field is obtained: [23]

$$\Box\Phi = \frac{dV}{d\Phi}. \tag{9}$$

To study gravitational waves, the linearized theory has to be analysed, with a little perturbation of the background, which is assumed given by a near Minkowskian background, i.e. a Minkowskian background plus $\Phi = \Phi_0$ (the Ricci scalar is assumed constant in the background) [9, 19]. Φ_0 is also assumed to be a steady minimum for the effective potential V, that called V_0. This assumption is vital for the further calculations [41] and its physical justification arises from the fact that the effective Φ scalar field is a function of the Ricci curvature and here the linearized theory is developed, i.e. only weak perturbations near a *fixed* curvature are considered. Thus, such a minimum has to be *steady*. This is an analysis *totally equivalent* to the case of the linearization process for scalar-tensor gravity, see for example [16]. In Section 4 of [16] the authors claim that *"we linearize the equations near the background ($\eta_{\mu\nu}$, φ_0), where φ_0 is a minimum of V"*. The difference with the present analysis is that in [16] the scalar field which is a minimum of the potential arises directly from the Brans-Dicke theory [23], while in the present analysis an effective scalar field and an effective potential arise directly from spacetime curvature (the effective scalar field is the prime derivative $f'(R)$, see eq. (8)).

Thus, the potential presents a square (i.e. parabolic) trend, in function of the effective scalar field, near the minimum [41], i.e.

$$V \simeq V_0 + \frac{1}{2}\alpha\delta\Phi^2 \Rightarrow \frac{dV}{d\Phi} \simeq m^2\delta\Phi, \tag{10}$$

and the constant m has mass dimension.

Putting

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$\Phi = \Phi_0 + \delta\Phi. \tag{11}$$

to first order in $h_{\mu\nu}$ and $\delta\Phi$, calling $\widetilde{R}_{\mu\nu\rho\sigma}$, $\widetilde{R}_{\mu\nu}$ and $\widetilde{R}$ the linearized quantities which correspond to $R_{\mu\nu\rho\sigma}$, $R_{\mu\nu}$ and R, the linearized field equations are obtained [12, 19, 31]:

$$\widetilde{R}_{\mu\nu} - \frac{\widetilde{R}}{2}\eta_{\mu\nu} = (\partial_\mu\partial_\nu h_f - \eta_{\mu\nu}\Box h_f)$$

$$\Box h_f = m^2 h_f, \tag{12}$$

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where

$$h_f \equiv \frac{\delta\Phi}{\Phi_0}. \tag{13}$$

Then, from the second of eqs. (12), the mass can be defined like

$$m \equiv \sqrt{\frac{\Box h_f}{h_f}} = \sqrt{\frac{\Box\delta\Phi}{\delta\Phi}}. \tag{14}$$

Thus, as the mass is generated by variation of a function of the Ricci scalar, in a certain sense, it is generated by variation of spacetime curvature. In this way, the theory is totally generalized for an arbitrary f function of the Ricci scalar, improving the results in [9, 19, 40], where only particular theories have been discussed..

$\widetilde{R}_{\mu\nu\rho\sigma}$ and eqs. (12) are invariants for gauge transformations [9, 12, 19, 40]

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} - \partial_{(\mu}\epsilon_{\nu)}$$

$$\delta\Phi \to \delta\Phi' = \delta\Phi; \tag{15}$$

then

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{h}{2}\eta_{\mu\nu} + \eta_{\mu\nu}h_f \tag{16}$$

can be defined, and, considering the transform for the parameter ϵ^μ

$$\Box\epsilon_\nu = \partial^\mu\bar{h}_{\mu\nu}, \tag{17}$$

a gauge parallel to the Lorenz one of electromagnetic waves can be chosen:

$$\partial^\mu\bar{h}_{\mu\nu} = 0. \tag{18}$$

In this way,the field equations read like

$$\Box\bar{h}_{\mu\nu} = 0 \tag{19}$$

$$\Box h_f = m^2 h_f \tag{20}$$

Solutions of eqs. (19) and (20) are plan waves [12, 19]:

$$\bar{h}_{\mu\nu} = A_{\mu\nu}(\overrightarrow{p})\exp(ip^\alpha x_\alpha) + c.c. \tag{21}$$

$$h_f = a(\overrightarrow{p})\exp(iq^\alpha x_\alpha) + c.c. \tag{22}$$

where

$$k^\alpha \equiv (\omega, \overrightarrow{p}) \qquad \omega = p \equiv |\overrightarrow{p}|$$

$$q^\alpha \equiv (\omega_m, \overrightarrow{p}) \qquad \omega_m = \sqrt{m^2 + p^2}. \tag{23}$$

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In eqs. (19) and (21) the equation and the solution for the standard waves of General Relativity [31, 32] have been obtained, while eqs. (20) and (22) are respectively the equation and the solution for the massive mode (see also [9, 12, 19, 40]).

The fact that the dispersion law for the modes of the massive field h_f is not linear has to be emphasized. The velocity of every "ordinary" (i.e. which arises from General Relativity) mode $\bar{h}_{\mu\nu}$ is the light speed c, but the dispersion law (the second of eq. (23)) for the modes of h_f is that of a massive field which can be discussed like a wave-packet [9, 12, 19, 40]. Also, the group-velocity of a wave-packet of h_f centred in $\overrightarrow{p}$ is

$$\overrightarrow{v_G} = \frac{\overrightarrow{p}}{\omega}, \tag{24}$$

which is exactly the velocity of a massive particle with mass m and momentum $\overrightarrow{p}$.

From the second of eqs. (23) and eq. (24) it is simple to obtain:

$$v_G = \frac{\sqrt{\omega^2 - m^2}}{\omega}. \tag{25}$$

Then, as one wants a constant speed of the wave-packet, it has to be [9, 12, 19]

$$m = \sqrt{(1 - v_G^2)}\omega. \tag{26}$$

Now, the analysis can remain in the Lorenz gauge with transformations of the type $\Box\epsilon_\nu = 0$; this gauge gives a condition of transverse effect for the ordinary part of the field: $k^\mu A_{\mu\nu} = 0$, but does not give the transverse effect for the total field $h_{\mu\nu}$. From eq. (16) it is

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{\bar{h}}{2}\eta_{\mu\nu} + \eta_{\mu\nu}h_f. \tag{27}$$

At this point, if being in the massless case [9, 12, 19], one puts

$$\Box\epsilon^\mu = 0$$

$$\partial_\mu\epsilon^\mu = -\frac{\bar{h}}{2} + h_f, \tag{28}$$

which gives the total transverse effect of the field. But in the massive case this is impossible. In fact, by applying the Dalembertian operator to the second of eqs. (28) and by using the field equations (19) and (20) it is

$$\Box\epsilon^\mu = m^2 h_f, \tag{29}$$

which is in contrast with the first of eqs. (28). In the same way, it is possible to show that it does not exist any linear relation between the tensor field $\bar{h}_{\mu\nu}$ and the massive field h_f. Thus, a gauge in which $h_{\mu\nu}$ is purely spatial cannot be

chosen (i.e. $h_{\mu 0} = 0$ cannot be put, see eq. (27)) . But the traceless condition to the field $\bar{h}_{\mu\nu}$ can be put :

$$\Box \epsilon^\mu = 0$$

$$\partial_\mu \epsilon^\mu = -\frac{\bar{h}}{2}.$$

(30)

These equations imply

$$\partial^\mu \bar{h}_{\mu\nu} = 0.$$

(31)

To save the conditions $\partial_\mu \bar{h}^{\mu\nu}$ and $\bar{h} = 0$ transformations like

$$\Box \epsilon^\mu = 0$$

$$\partial_\mu \epsilon^\mu = 0$$

(32)

can be used and, taking $\overrightarrow{p}$ in the z direction, a gauge in which only A_{11}, A_{22}, and $A_{12} = A_{21}$ are different to zero can be chosen. The condition $\bar{h} = 0$ gives $A_{11} = -A_{22}$. Now, putting these equations in eq. (27), it is

$$h_{\mu\nu}(t, z) = A^+(t - z)e_{\mu\nu}^{(+)} + A^\times(t - z)e_{\mu\nu}^{(\times)} + h_f(t - v_G z)\eta_{\mu\nu}.$$

(33)

The term $A^+(t - z)e_{\mu\nu}^{(+)} + A^\times(t - z)e_{\mu\nu}^{(\times)}$ describes the two standard polarizations of gravitational waves which arise from General Relativity, while the term $h_f(t - v_G z)\eta_{\mu\nu}$ is the massive field arising from the high order theory. In other words, the function f' of the Ricci scalar generates a third massive polarization for gravitational waves which is not present in standard General Relativity.

4   The primordial production of the third polarization

Now, let us consider the primordial physical process, which gave rise to a characteristic spectrum Ω_{gw} for relic GWs. Such physical process has been analysed in different ways: respectively in refs. [25, 26] and [27, 28] but only for the components of eq. (33) which arises from General Relativity. In [29] the process has been extended to scalar-tensor gravity. Actually, the process can be further improved showing the primordial production of the third polarization of eq. (33).

Before starting with the analysis, let us recall that, considering a stochastic background of GWs, it can be characterized by a dimensionless spectrum [25, 26, 27, 28, 29]

$$\Omega_{gw}(f) \equiv \frac{1}{\rho_c}\frac{d\rho_{gw}}{d\ln f},$$

(34)

where

$$\rho_c \equiv \frac{3H_0^2}{8G}$$

(35)

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is the (actual) critical density energy, ρ_c of the Universe, H_0 the actual value of the Hubble expansion rate and $d\rho_{gw}$ the energy density of relic GWs in the frequency range f to $f + df$.

The existence of a relic stochastic background of GWs arises from generals assumptions, i.e. from a mixing between basic principles of classical theories of gravity and of quantum field theory. The strong variations of the gravitational field in the early universe amplify the zero-point quantum oscillations and produce relic GWs. It is well known that the detection of relic GWs is the only way to learn about the evolution of the very early universe, up to the bounds of the Planck epoch and the initial singularity [21, 25, 26, 27, 28, 29]. It is very important to stress the unavoidable and fundamental character of this mechanism. The model derives from the inflationary scenario for the early universe [35, 36], which is tuned in a good way with the WMAP data on the Cosmic Background Radiation (CBR) (in particular exponential inflation and spectral index ≈ 1 [37, 38]). Inflationary models of the early Universe were analysed in the early and middles 1980's (see [35] for a review), starting from an idea of A. Guth [36]. These are cosmological models in which the Universe undergoes a brief phase of a very rapid expansion in early times. In this context, the expansion could be power-law or exponential in time. Inflationary models provide solutions to the horizon and flatness problems and contain a mechanism which creates perturbations in all fields. Important for our goals is that this mechanism also provides a distinctive spectrum of relic GWs. The GWs perturbations arise from the uncertainty principle and the spectrum of relic GWs is generated from the adiabatically-amplified zero-point fluctuations [21, 25, 26, 27, 28, 29].

Now, the calculation for a simple inflationary model will be shown for the third polarization of eq. (33), following the works of Allen [25, 26] that performed the calculation in the case of standard General Relativity and Corda, Capozziello and De Laurentis [29, 30] that extended the process to scalar GWs. In a recent paper [21], such a process has been applied to the $f(R)$ theories arising from the action (1). But, in [21] a different point of view has been considered. In that case, using a conform analysis, the authors discussed such a process in respect to the two standard polarizations which arises from standard General Relativity. In the following the analysis is focused to the third massive polarization. Thus, in a certain sense, one can say that the present analysis is an integration of the analysis in [21].

It will be assumed that the universe is described by a simple cosmology in two stages, an inflationary De Sitter phase and a radiation dominated phase [21, 25, 26, 27, 28, 29]. The line element of the spacetime is given by

$$ds^2 = a^2(\eta)[-d\eta^2 + d\overrightarrow{x}^2 + h_{\mu\nu}(\eta, \overrightarrow{x})dx^\mu dx^\nu]. \tag{36}$$

(36) has to be a solution of the general field equations (6). In fact, even if such a form is allowed, it could have absolutely different behavior (see [44, 45, 46]). For instance one needs to show that inflation is present in the proposed model. But, in the case of f(R) theories, which are the ones that we are treating here, both of the two conditions are, in general, satisfied. In fact, we recall that the

original inflation was proposed by Starobinsky in the classical papers [48, 49] by using the simplest f(R) theory, i.e. the R^2 one. On the other hand, the potential presence and the importance of standard De Sitter inflation in the general framework of the primordial production of relic gravitational waves has been recently shown in [50] considering a different point of view. In that case, using a conform analysis, the authors discussed such a process in respect to the two standard polarizations which arises from standard General Relativity. In the following, the analysis is focused to the third massive polarization. Thus, in a certain sense, the present analysis is an integration of the analysis in [50]. Remaining in the tapestry of f(R) theories, there are lots of examples in the literature where the line element (36), which is the perturbed form of the standard conformally flat Robertson-Walker one, results a solution of the general field equations (6). One can see for example the recent reviews [51, 52, 53].

In the line element (36), by considering only the third polarization, the metric perturbation (33) reduces to

$$h_{\mu\nu} = h_f I_{\mu\nu}, \tag{37}$$

where

$$I_{\mu\nu} \equiv \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1. \end{matrix} \tag{38}$$

In the De Sitter phase ($\eta < \eta_1$) the equation of state is $P = -\rho = const$, the scale factor is $a(\eta) = \eta_1^2 \eta_0^{-1} (2\eta_1 - \eta)^{-1}$ and the Hubble constant is given by $H(\eta) = H_{ds} = c\eta_0/\eta_1^2$.

In the radiation dominated phase ($\eta > \eta_1$) the equation of state is $P = \rho/3$, the scale factor is $a(\eta) = \eta/\eta_0$ and the Hubble constant is given by $H(\eta) = c\eta_0/\eta^2$ [21, 25, 26, 29, 30].

Expressing the scale factor in terms of comoving time defined by

$$cdt = a(t)d\eta \tag{39}$$

one gets

$$a(t) \propto \exp(H_{ds}t) \tag{40}$$

during the De Sitter phase and

$$a(t) \propto \sqrt{t} \tag{41}$$

during the radiation dominated phase. The horizon and flatness problems are solved if [35, 36]

$$\frac{a(\eta_0)}{a(\eta_1)} > 10^{27}$$

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The third polarization generates weak perturbations $h_{\mu\nu}(\eta, \overrightarrow{x})$ of the metric (37) that can be written, in terms of the conformal time η, in the form

$$h_{\mu\nu} = I_{\mu\nu}(\hat{k})X(\eta)\exp(\overrightarrow{k} \cdot \overrightarrow{x}), \tag{42}$$

where $\overrightarrow{k}$ is a constant wavevector and

$$h_f(\eta, \overrightarrow{k}, \overrightarrow{x}) = X(\eta)\exp(\overrightarrow{k} \cdot \overrightarrow{x}). \tag{43}$$

By putting $Y(\eta) = a(\eta)X(\eta)$ one performs the standard linearized calculation in which the connections (i.e. the Cristoffel coefficients), the Riemann tensor, the Ricci tensor and the Ricci scalar curvature are computed. Then, from the Friedman linearized equations, the function $Y(\eta)$ satisfies the equation

$$Y'' + (|\overrightarrow{k}|^2 - \frac{a''}{a})Y = 0 \tag{44}$$

where $'$ denotes derivative with respect to the conformal time. Clearly, this is the equation for a parametrically disturbed oscillator.

The solutions of eq. (44) give the solutions for the function $X(\eta)$, that can be expressed in terms of elementary functions simple cases of half integer Bessel or Hankel functions [21, 25, 26, 29, 30] in both of the inflationary and radiation dominated eras:

For $\eta < \eta_1$

$$X(\eta) = \frac{a(\eta_1)}{a(\eta)}[1 + H_{ds}\omega^{-1}]\exp -ik(\eta - \eta_1), \tag{45}$$

for $\eta > \eta_1$

$$X(\eta) = \frac{a(\eta_1)}{a(\eta)}[\alpha \exp -ik(\eta - \eta_1) + \beta \exp ik(\eta - \eta_1), \tag{46}$$

where $\omega = ck/a$ is the angular frequency of the wave (that is function of the time because of the constance of $k = |\overrightarrow{k}|$), α and β are time-independent constants which can be obtained demanding that both X and $dX/d\eta$ are continuous at the boundary $\eta = \eta_1$ between the inflationary and the radiation dominated eras of the cosmological expansion. With this constrain it is

$$\alpha = 1 + i\frac{\sqrt{H_{ds}H_0}}{\omega} - \frac{H_{ds}H_0}{2\omega^2} \tag{47}$$

$$\beta = \frac{H_{ds}H_0}{2\omega^2} \tag{48}$$

In eqs. (47), (48) $\omega = ck/a(\eta_0)$ is the angular frequency that would be observed today. Calculations like this are referred in the literature as Bogoliubov coefficient methods [21, 25, 26, 29, 30].

As inflation damps out any classical or macroscopic perturbations, the minimum allowed level of fluctuations is that required by the uncertainty principle. The solution (45) corresponds precisely to this De Sitter vacuum state [21, 25, 26, 29, 30]. Then, if the period of inflation was long enough, the observable properties of the Universe today should be the same properties of a Universe started in the De Sitter vacuum state.

In the radiation dominated phase the coefficients of α are the eigenmodes which describe particles while the coefficients of β are the eigenmodes which describe antiparticles. Thus, the number of created particles of angular frequency ω in this phase is

$$N_\omega = |\beta_\omega|^2 = (\frac{H_{ds}H_0}{2\omega^2})^2. \tag{49}$$

Now, one can write an expression for the energy spectrum of the relic gravitational waves background in the frequency interval $(\omega, \omega + d\omega)$ as

$$d\rho_{gw} = 2\hbar\omega(\frac{\omega^2 d\omega}{2\pi^2 c^3})N_\omega = \frac{\hbar H_{ds}^2 H_0^2}{4\pi^2 c^3}\frac{d\omega}{\omega} = \frac{\hbar H_{ds}^2 H_0^2}{4\pi^2 c^3}\frac{df}{f}. \tag{50}$$

Eq. (50) can be rewritten in terms of the present day and the De Sitter energy-density of the Universe. The Hubble expansion rates is

$$H_0^2 = \frac{8\pi G\rho_c}{3}, \ H_{ds}^2 = \frac{8\pi G\rho_{ds}}{3}.$$

Then, introducing the Planck density

$$\rho_{Planck} \equiv \frac{c^7}{\hbar G^2} \tag{51}$$

the spectrum is

$$\Omega_{gw}(f) = \frac{1}{\rho_c}\frac{d\rho_{sgw}}{d\ln f} = \frac{f}{\rho_c}\frac{d\rho_{gw}}{df} = \frac{16}{9}\frac{\rho_{ds}}{\rho_{Planck}}. \tag{52}$$

Some comments are needed. The computation works for a very simplified model that does not include the matter dominated era. Including this era, the redshift has to be considered. An enlightening computation parallel to the one in [26] gives

$$\Omega_{gw}(f) = \frac{16}{9}\frac{\rho_{ds}}{\rho_{Planck}}(1 + z_{eq})^{-1}, \tag{53}$$

for the waves which at the time in which the Universe was becoming matter dominated had a frequency higher than H_{eq}, the Hubble constant at that time. This corresponds to frequencies $f > (1 + z_{eq})^{1/2}H_0$, where z_{eq} is the redshift of the Universe when the matter and radiation energy density were equal. The redshift correction in equation (53) is needed because the Hubble parameter, which is governed by Friedman equations, should be different from the observed one H_0 for a Universe without matter dominated era.

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At lower frequencies the spectrum is [21, 25, 26, 29, 30]

$$\Omega_{gw}(f) \propto f^{-2}. \tag{54}$$

Moreover, the results (52) and (53), which are not frequency dependent, cannot be applied to all the frequencies. For waves with frequencies less than H_0 today, the energy density cannot be defined, because the wavelength becomes longer than the Hubble radius. In the same way, at high frequencies there is a maximum frequency above which the spectrum drops to zero rapidly. In the above computation it has been implicitly assumed that the phase transition from the inflationary to the radiation dominated epoch is instantaneous. In the real Universe this phase transition occurs over some finite time $\Delta\tau$, and above a frequency

$$f_{max} = \frac{a(t_1)}{a(t_0)} \frac{1}{\Delta\tau}, \tag{55}$$

which is the redshifted rate of the transition, Ω_{gw} drops rapidly. These two cutoffs, at low and high frequencies, to the spectrum force the total energy density of the relic gravitational waves to be finite. For GUT energy-scale inflation it is [21, 25, 26, 29, 30].

$$\frac{\rho_{ds}}{\rho_{Planck}} \approx 10^{-12}. \tag{56}$$

5 Tuning with WMAP data

It is well known that WMAP observations put strongly severe restrictions on the spectrum of relic gravitational waves. In fig. 1 the spectrum Ω_{gw} is mapped following [20]: the amplitude is chosen (determined by the ratio $\frac{\rho_{ds}}{\rho_{Planck}}$) to be *as large as possible, consistent with the WMAP constraints on tensor perturbations.* Nevertheless, because the spectrum falls off $\propto f^{-2}$ at low frequencies, this means that today, at LIGO-Virgo and LISA frequencies (indicate by the lines in fig. 1) [20], it is

$$\Omega_{gw}(f)h_{100}^2 < 9 * 10^{-13}. \tag{57}$$

Let us calculate the correspondent strain at $\approx 100Hz$, where interferometers like Virgo and LIGO have a maximum in sensitivity. The well known equation for the characteristic amplitude, adapted for the third component of GWs can be used [20]:

$$h_{fc}(f) \simeq 1.26 * 10^{-18} (\frac{1Hz}{f}) \sqrt{h_{100}^2 \Omega_{gw}(f)}, \tag{58}$$

obtaining [20]

$$h_{fc}(100Hz) < 1.7 * 10^{-26}. \tag{59}$$

Then, as we expect a sensitivity order of 10^{-22} for interferometers at $\approx 100Hz$, four order of magnitude have to be gained. Let us analyse smaller frequencies too. The sensitivity of the Virgo interferometer is of the order of 10^{-21} at $\approx 10Hz$ and in that case it is [20]

$$h_{fc}(10Hz) < 1.7 * 10^{-25}. \tag{60}$$

The sensitivity of the LISA interferometer will be of the order of 10^{-22} at $10^{-3} \approx Hz$ and in that case it is [20]

$$h_{fc}(100Hz) < 1.7 * 10^{-21}. \tag{61}$$

Then, a stochastic background of relic gravitational waves could be, in principle, detected by the LISA interferometer.

We emphasize the sentence *in principle*. Actually, one has to take into account the Galactic confusion background which will dominate over the instrumental noise. Thus, it is still questionable whether the relic gravitons produced in GR could be detected by LISA [41].

On the other hand, the assumption that all the tensor perturbations in the Universe are due to a stochastic background of GWs is quit strong, but the results (57), (59), (60) and (61) can be considered like upper bounds.

Figure 1, which is adapted from ref. [20], shows that the spectrum of relic SGWs in inflationary models is flat over a wide range of frequencies. The horizontal axis is $\log_{10}$ of frequency, in Hz. The vertical axis is $\log_{10} \Omega_{gsw}$. The inflationary spectrum rises quickly at low frequencies (wave which re-entered in the Hubble sphere after the Universe became matter dominated) and falls off above the (appropriately redshifted) frequency scale f_{max} associated with the fastest characteristic time of the phase transition at the end of inflation. The amplitude of the flat region depends only on the energy density during the inflationary stage; we have chosen the largest amplitude consistent with the WMAP constrains on tensor perturbations. This means that at LIGO and LISA frequencies, $\Omega_{gw}(f)h_{100}^2 < 9 * 10^{-13}$.

6 Potential detection with interferometers

Before starting the discussion of the potential interferometric detection of the massive GWs polarization, there is another point which has to be clarified. The attentive reader [41] asks what would happen to Hulse-Taylor pulsar [54]. As it is clear from eq. (33), by fixing the massless component of the metric to GR value, the third term should carry the additional energy which should affect evolution of the binary system [41]. This problem has been discussed by Shibata, Nakao and Nakamura in [15]. In such a work, it has been shown that the energy associated to the monopole mode of the scalar particle is quite lower in respect to the energy carried by ordinary quadrupole modes arising from
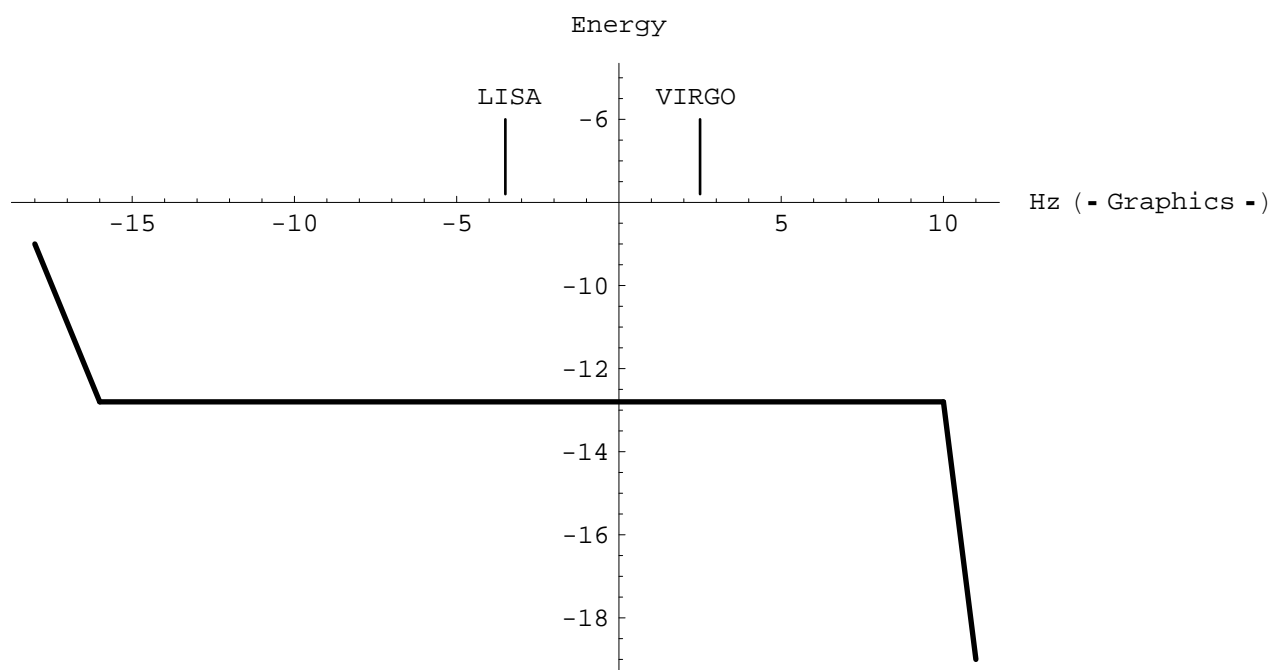
Figure 1: The spectrum of relic SGWs in inflationary models is flat over a wide range of frequencies. The horizontal axis is $\log_{10}$ of frequency, in Hz. The vertical axis is $\log_{10}\Omega_{gsw}$.

the linearization of standard General Relativity. Thus, in these conditions, the evolution of the binary system should be not affected by the third polarization if one remains into observing error bars.

On the other and, even if the ratio of differential mode to common mode is large in the case of the binary pulsar, it could be, in principle, similar and potentially small in cosmology. In fact, in the case of $f(R)$ theories such a mode arises directly from a function of the Ricci scalar, see the first identification (8), i.e. *it arises from spacetime curvature*, and it is well known that, being the production of relic GWs near the bounds of the Planck epoch and the initial singularity, in this early Era both of spacetime curvature and its variations were very high. Thus, in this cosmological case, the amplitude of the common mode could be even dominant in respect to differential modes.

After these clarifications, let us start the discussion regarding the potential detection.

Even if eqs. (59) and (60) show that the detection of relic GWs is quite difficult, people hope in better sensitivity of advanced projects. In this case, it is interesting to discuss the interaction between interferometers and massive GWs.

Considering only the third polarization $h_f(t - v_G z)\eta_{\mu\nu}$ the line element associated to eq. (33) becomes the conformally flat one

$$ds^2 = [1 + h_f(t, z)](-dt^2 + dz^2 + dx^2 + dy^2). \tag{62}$$

As the analysis on the motion of test masses is performed in a laboratory environment on Earth, the coordinate system in which the space-time is locally flat is typically used and the distance between any two points is given simply by the difference in their coordinates in the sense of Newtonian physics [12, 13, 14, 16, 31]. This frame is the proper reference frame of a local observer, located for example in the position of the beam splitter of an interferometer. In this frame GWs manifest themself by exerting tidal forces on the masses (the mirror and the beam-splitter in the case of an interferometer). A detailed analysis of the frame of the local observer is given in ref. [31], sect. 13.6. Here only the more important features of this coordinate system are recalled:

the time coordinate x_0 is the proper time of the observer O;

spatial axes are centred in O;

in the special case of zero acceleration and zero rotation the spatial coordinates x_j are the proper distances along the axes and the frame of the local observer reduces to a local Lorentz frame: in this case the line element reads [31]

$$ds^2 = -(dx^0)^2 + \delta_{ij}dx^i dx^j + O(|x^j|^2)dx^\alpha dx^\beta. \tag{63}$$

The effect of the gravitational wave on test masses is described by the equation

$$\ddot{x}^i = -\widetilde{R}^i_{0k0}x^k, \tag{64}$$

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which is the equation for geodesic deviation in this frame.

Thus, to study the effect of the massive gravitational wave on test masses, $\widetilde{R}^i_{0k0}$ have to be computed in the proper reference frame of the local observer. But, because the linearized Riemann tensor $\widetilde{R}_{\mu\nu\rho\sigma}$ is invariant under gauge transformations [9, 12, 13, 14, 31], it can be directly computed from eq. (37).

From [31] it is:

$$\widetilde{R}_{\mu\nu\rho\sigma} = \frac{1}{2}\{\partial_\mu\partial_\beta h_{\alpha\nu} + \partial_\nu\partial_\alpha h_{\mu\beta} - \partial_\alpha\partial_\beta h_{\mu\nu} - \partial_\mu\partial_\nu h_{\alpha\beta}\}, \tag{65}$$

that, in the case eq. (37), begins

$$\widetilde{R}^\alpha_{0\gamma0} = \frac{1}{2}\{\partial^\alpha\partial_0 h_f \eta_{0\gamma} + \partial_0\partial_\gamma h_f \delta^\alpha_0 - \partial^\alpha\partial_\gamma h_f \eta_{00} - \partial_0\partial_0 h_f \delta^\alpha_\gamma\}; \tag{66}$$

the different elements are (only the non zero ones will be written):

$$\partial^\alpha\partial_0 h_f \eta_{0\gamma} = \left\{ \begin{array}{ll} \partial_t^2 h_f & for \quad \alpha = \gamma = 0 \\[2mm] -\partial_z\partial_t h_f & for \quad \alpha = 3; \gamma = 0 \end{array} \right\} \tag{67}$$

$$\partial_0\partial_\gamma h_f \delta^\alpha_0 = \left\{ \begin{array}{ll} \partial_t^2 h_f & for \quad \alpha = \gamma = 0 \\[2mm] \partial_t\partial_z h_f & for \quad \alpha = 0; \gamma = 3 \end{array} \right\} \tag{68}$$

$$- \partial^\alpha\partial_\gamma h_f \eta_{00} = \partial^\alpha\partial_\gamma h_f = \left\{ \begin{array}{ll} -\partial_t^2 h_f & for \quad \alpha = \gamma = 0 \\[2mm] \partial_z^2 h_f & for \quad \alpha = \gamma = 3 \\[2mm] -\partial_t\partial_z h_f & for \quad \alpha = 0; \gamma = 3 \\[2mm] \partial_z\partial_t h_f & for \quad \alpha = 3; \gamma = 0 \end{array} \right\} \tag{69}$$

$$- \partial_0\partial_0 h_f \delta^\alpha_\gamma = -\partial_z^2 h_f \quad for \quad \alpha = \gamma . \tag{70}$$

Now, putting these results in eq. (66) one obtains:

$$\widetilde{R}^1_{010} = -\tfrac{1}{2}\ddot{h}_f$$

$$\widetilde{R}^2_{010} = -\tfrac{1}{2}\ddot{h}_f \tag{71}$$

$$\widetilde{R}^3_{030} = \tfrac{1}{2}\Box h_f.$$

But, putting the second of equations (12) in the third of eqs. (71), it is

$$\widetilde{R}^3_{030} = \frac{1}{2}m^2 h_f, \tag{72}$$

which shows that the field is not transversal.

In fact, using eq. (64) it results

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$$\ddot{x} = \frac{1}{2}\ddot{h}_f x, \tag{73}$$

$$\ddot{y} = \frac{1}{2}\ddot{h}_f y \tag{74}$$

and

$$\ddot{z} = -\frac{1}{2}m^2 h_f(t,z)z. \tag{75}$$

Then, the effect of the mass is the generation of a *longitudinal* force (in addition to the transverse one).

For a better understanding of this longitudinal force, let us analyse the effect on test masses in the context of the geodesic deviation.

Following [14] one puts

$$\widetilde{R}^i_{0j0} = \frac{1}{2} \begin{pmatrix} -\partial_t^2 & 0 & 0 \\ 0 & -\partial_t^2 & 0 \\ 0 & 0 & m^2 \end{pmatrix} h_f(t,z) = -\frac{1}{2}T_{ij}\partial_t^2 h_f + \frac{1}{2}L_{ij}m^2 h_f. \tag{76}$$

Here the transverse projector with respect to the direction of propagation of the GW $\widehat{n}$, defined by

$$T_{ij} = \delta_{ij} - \widehat{n}_i\widehat{n}_j, \tag{77}$$

and the longitudinal projector defined by

$$L_{ij} = \widehat{n}_i\widehat{n}_j \tag{78}$$

have been used. In this way, the geodesic deviation equation (64) can be rewritten like

$$\frac{d^2}{dt^2}x_i = \frac{1}{2}\partial_t^2 h_f T_{ij}x_j - \frac{1}{2}m^2 h_f L_{ij}x_j. \tag{79}$$

Thus, it appears clear that the effect of the mass present in the GW generates a longitudinal force proportional to m^2 which is in addition to the transverse one. But if $v(\omega) \to 1$ in eq. (25) one gets $m \to 0$, and the longitudinal force vanishes. Then, it is clear that the longitudinal mode arises from the fact that the GW does no propagate at the speed of light.

Now, let us analyse the detectability of the third polarization computing the pattern function of a detector to this massive component. One has to recall that it is possible to associate to a detector a *detector tensor* that, for an interferometer with arms along the $\hat{u}$ e $\hat{v}$ directions in respect to the propagating gravitational wave (see figure 2), is defined by [14]

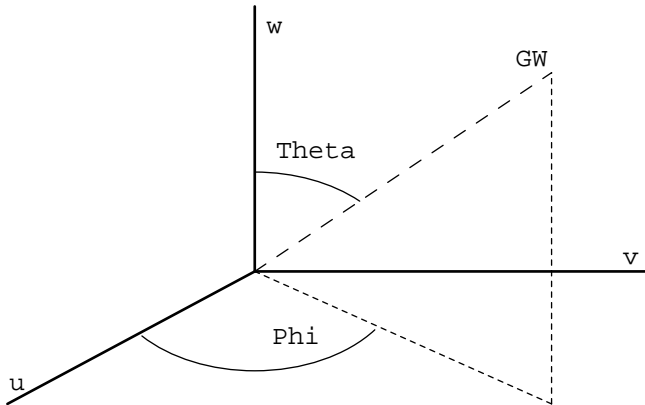$$D^{ij} \equiv \frac{1}{2}(\hat{v}^i\hat{v}^j - \hat{u}^i\hat{u}^j). \tag{80}$$

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Figure 2: A gravitational wave propagating in the z direction

If the detector is an interferometer [1, 2, 3, 4, 5, 6, 7, 8, 21], the signal induced by a GW of a generic polarization, here labelled with $s(t)$, is the phase shift, which is proportional to [14]

$$s(t) \sim D^{ij} \widetilde{R}_{i0j0} \tag{81}$$

and, using equations (76), one gets

$$s(t) \sim -\sin^2 \theta \cos 2\phi. \tag{82}$$

The angular dependence (82), is different from the two well known standard ones arising from general relativity which are, respectively $(1 + \cos^2 \theta) \cos 2\phi$ for the + polarization and $-\cos \theta \sin 2\vartheta$ for the $\times$ polarization. Thus, in principle, the angular dependence (82) could be used to discriminate among $f(R)$ theories and general relativity, if present or future detectors will achieve a high sensitivity. The third angular dependence is shown in figure.

For a sake of completeness, let us recall that there is one more problem with the potential detection of the common mode. This is because the sensitivity curve drawn for different detectors is for the differential mode. The common mode is also registered during the GW experiment but it is much noisier, therefore it has different sensitivity level [41]. Actually, this is correct for studies on potential detection in the case of wavelength of the wave much larger than the linear dimension of the interferometer (low frequencies approximation). In previous analysis, we have implicitly assumed to be in such an approximation, and in particular, eqs. (80), (81) and (82) are strictly valid only in this approximation. A recent analysis [55] has shown that, in the case in which the differential mode is massive, the response function of an interferometer *increases* with increasing frequency differently from the case of massless modes where the response function *decreases* with increasing frequency. This opens important perspectives for a potential detection of the massive mode at high frequency. In
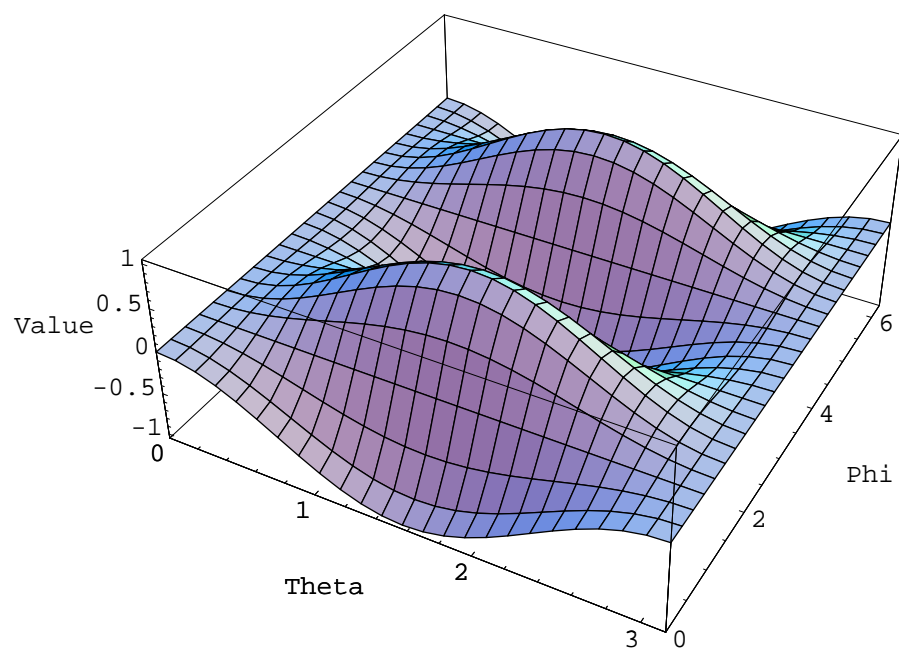
Figure 3: The (dimensionless) angular dependence (82)

[55] the response function has only been computed in the case of a massive mode propagating parallel to one arm of the interferometer, thus, further studies in this direction are needed. For example, it will be quite interesting to generalize the frequency dependence of the angular pattern (82).

7 Conclusion remarks

It has been shown that, in general, $f(R)$ theories produce a third massive polarization of gravitational waves and the primordial production of this polarization has been analysed adapting the adiabatically-amplified zero-point fluctuations process at this case and generalizing previous results in which only particular cases have been discussed.

The presence of the mass could also have important applications in cosmology because the fact that gravitational waves can have mass could give a contribution to the dark matter of the Universe.

An upper bound for these relic gravitational waves, which arises from the WMAP constrains, has been also released and at the end of the paper and the potential detection of such massive GWs using interferometers like Virgo and LIGO has been discussed.

Acknowledgements

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